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Solitary wave solutions in double sinh-Gordon system

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Abstract. Two new classes of solitary wave solutions for the recently suggested double sinh-Gordon system have been obtained. These solutions possess nice stability and asymptotic properties and are characterised by zero topological charge. The existence and behaviour of these solutions in more than two space–time dimensions and the properties of multisolitary waves have been studied in detail.

1. Introduction

The sine-Gordon (SG) equation has a wide range of applications in solid state physics and nonlinear optics. In $(1+1)$ dimensions the SG field system undergoes a second-order phase transition (Babu Joseph and Kuriakose 1982). A number of related nonlinear models have been analysed in the recent past which include the sinh-Gordon (ShG) (Mckean 1981) and the various double sine-Gordon (DSG) equations (Bullough *et al* 1980). The ShG system arises as a trivial map of the SG field, but the ShG system differs from the latter by the absence of soliton solutions (Ablowitz *et al* 1973). Various kink (Bullough *et al* 1980) and 'soliton' solutions (Burt 1978) of the DSG system have also been considered. The DSG equation arises, for instance, in the treatment of quasi-one-dimensional charge-density wave condensates of organic linear conductors like TTF-TCNQ (Rice 1978).

A new member has recently been added to the DSG family of equations by Behera and Khare (1981). Because of its analogy with the DSG model, the Behera–Khare model may be called the double sinh-Gordon or DShG system, which is characterised by the potential

$$V(\phi)=\tfrac{1}{8}\eta^2\cosh 4\phi-\eta\cosh 2\phi-2\eta^2+4\eta \tag{1.1}$$

where η is a real parameter. They found a kink solution for this model and demonstrated the possibility of calculating the exact free energy associated with the second-order phase transition that the system undergoes. In this paper we obtain, using the methods of bilinear operators (Hirota 1972) and base equations (Burt 1978), two new classes of solitary wave solutions of the DShG system. Characterised as they are by a vanishing topological charge, these new solutions can be considered non-topological objects (Lee 1976). Application of the base equation technique leads to N-solitary wave solutions in space–time with dimension greater than two. The linear stability of the waves in $(1+1)$ dimensions is examined in detail and we find that the solutions are stable. The asymptotic stability of the N-solitary wave solution is also discussed. Despite the somewhat popular practice (Burt 1978) of considering asymptotically vanishing solutions generated by the base equation technique to be solitons, in the absence of any study of the collisional stability of the waves we do not claim that our solitary wave solutions are solitons.

2. Solitary waves by the bilinear operator method

The equation of motion corresponding to the potential defined in (1.1) is given by

$$\phi_{xx} - \phi_{tt} = \tfrac{1}{2}\eta^2 \sinh 4\phi - 2\eta \sinh 2\phi. \tag{2.1}$$

Define a transformation

$$\phi = \tanh^{-1}(g/f) \tag{2.2}$$

so that (2.1) yields the bilinear differential equation

$$(f^2+g^2)(D_x^2 - D_t^2)f\cdot g - f\cdot g(D_x^2 - D_t^2)(f\cdot f + g\cdot g)$$
$$= 2\eta^2 f\cdot g(f^2+g^2) - 4\eta(f^2-g^2)f\cdot g \tag{2.3}$$

where D_z^n is the bilinear differential operator (Hirota 1972) defined by

$$D_z^n a\cdot b \equiv [(\partial/\partial z) - (\partial/\partial z')]^n a(z)\cdot b(z')|_{z=z'}.$$

On decoupling (2.3), we find

$$(D_x^2 - D_t^2)f\cdot g = 2\eta(\eta-2)f\cdot g \tag{2.4}$$

$$(D_x^2 - D_t^2)(f\cdot f + g\cdot g) = -8\eta g\cdot g \tag{2.5}$$

where

$$D_x^2 f\cdot g = D_x(f_x\cdot g - f\cdot g_x) \tag{2.6}$$

$$D_x^2 f\cdot f = 2D_x(f_x\cdot f). \tag{2.7}$$

We introduce power series expansions for f and g in a parameter ε which is very close to unity:

$$f = 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4 + \ldots \tag{2.8}$$

$$g = \varepsilon g_1 + \varepsilon^2 g_2 + \ldots. \tag{2.9}$$

On equating the coefficients of same powers of ε we obtain a set of differential equations. By proper selection of g_1 and f_2 we have obtained an exact solution in the form

$$f = 1 - [g\cdot g/(2\eta - 4)] \tag{2.10}$$

where

$$g = e^\theta \quad \text{and} \quad \theta = kx - \omega t + \delta \tag{2.11}$$

and we have set $\varepsilon = 1$, following Hirota (1972).

The associated dispersion relation is

$$k^2 - \omega^2 = 2\eta(\eta - 2). \tag{2.12}$$

Equation (2.10) gives the solitary wave solution

$$\phi(\theta) = \tanh^{-1} e^\theta [1 - e^{2\theta}/(2\eta - 4)]^{-1}. \tag{2.13}$$

The only other known solution of the DShG equation is a kink (Behera and Khare 1981):

$$\phi(x,t)=\tanh^{-1}\{(1-\tfrac{1}{2}\eta)(1-\tfrac{1}{2}\eta^2)^{-1/2}\tanh\sqrt{2}(1-\tfrac{1}{4}\eta^2)^{-1/\sqrt{2}}(x-ut)/[m(c^2-u^2)]^{1/2}\} \tag{2.14}$$

which exists for the values $|\eta|<2$.

In contrast with the kink solution (2.13), the new solution is defined only for $\eta\in(0,2)$ and can readily be extended to arbitrary dimensions. We might expect to obtain the multisolitary wave solution by setting

$$g=\sum_{i=1}^{N}e^{\theta_i}. \tag{2.15}$$

However, the corresponding power series of the form (2.8) and (2.9) do not terminate; thus the bilinear operator method fails to provide any such solutions.

3. *N*-solitary wave solutions

A differential equation whose solution is used to solve another differential equation is called a base equation. This method has been exploited by Ried and Burt (1974) in obtaining several solitary wave solutions.

Let us rewrite (2.1) in arbitrary dimensions as

$$\partial_\mu\partial^\mu\phi=\tfrac{1}{2}\eta^2\sinh 4\phi-2\eta\sinh 2\phi \tag{3.1}$$

with $\mu=0,1,\ldots,n-1$. The transformation

$$\phi=\sinh^{-1}\psi \tag{3.2}$$

converts (3.1) into the form

$$(1+\psi^2)^{-1/2}\partial_\mu\partial^\mu\psi-(1+\psi^2)^{-3/2}\psi(\partial_\mu\psi\partial^\mu\psi)$$
$$-2\eta^2\psi(1+\psi^2)^{1/2}(1+2\psi^2)+4\eta\psi(1+\psi^2)^{1/2}=0. \tag{3.3}$$

The following nonlinear differential equation may be taken as the base equation corresponding to (3.3):

$$(\partial_\mu\psi)(\partial^\mu\psi)=(1+\psi^2)(4\eta-6\eta^2-D)\psi^2-(1+\psi^2)(4\eta^2+B)\psi^4. \tag{3.4}$$

ψ may be expressed as

$$\psi=uA^{-1/2} \tag{3.5}$$

where

$$A=(1-(Bu^2/8m^2))^2-(Cu^4/12D^2) \tag{3.6}$$

and u satisfies the equations

$$\partial_\mu\partial^\mu u+D^2u=0 \tag{3.7}$$

$$(\partial_\mu u)(\partial^\mu u)+D^2u^2=0. \tag{3.8}$$

Henceforth, the last two equations can be employed as base equations for solving (3.4). These equations admit a solution

$$u=a\,e^{\alpha kx}; \tag{3.9}$$

then

$$B = 8\eta - 8\eta^2 \qquad C = -6\eta^2 \qquad D = 4\eta - 2\eta^2$$
$$k = k_0, k_1, k_2, \ldots, k_{n-1} \qquad x = x_0, x_1, \ldots, x_{n-1} \tag{3.10}$$

giving

$$\psi = u\{1 - \tfrac{1}{2}[(1-2\eta)/(4-2\eta)]u^2 + (4-2\eta)^{-2}u^4\}^{-1/2}. \tag{3.11}$$

The parameter α evidently satisfies the condition

$$\alpha = [-(4\eta - 2\eta^2)/k^2]^{1/2} \tag{3.12}$$

so that $4\eta - 2\eta^2 < 0$.

All the solutions of equations (3.7) and (3.8) are automatically the solutions of the DShG equation. By the linear superposition principle

$$u = \sum_{i=1}^{N} a_i \exp(\alpha_i k_i x) \tag{3.13}$$

is also a solution of (3.7). On substituting this solution in (3.11) an N-solitary wave solution of the DShG equation emerges with the additional conditions

$$\alpha_i \alpha_j \cdot k_i k_j + (4\eta - 2\eta^2)^2 = 0 \tag{3.14}$$

where

$$k_i = (k_{i0}, k_{i1}, \ldots, k_{in-1}) = (k_{i0}, \boldsymbol{K}_i)$$
$$k_i \neq k_j \qquad \boldsymbol{K}_i \cdot \boldsymbol{K}_j \neq 0$$

for any i, j. Equation (3.14) shows that the dimensionality of the space–time n and the maximum possible number of N-solitary waves are related by

$$N \leqslant 2n - 1. \tag{3.15}$$

Nevertheless, in $(1+1)$ dimensions only one solitary wave can be formed as there is only one independent wavevector and any other wavevector is necessarily parallel to it.

4. Linear stability of solutions in (1+1) dimensions

The static form of (2.13) is written

$$\phi_s(x) = \tanh^{-1} e^{kx}[1 - e^{2kx}/(2\eta - 4)]^{-1}. \tag{4.1}$$

Let

$$\phi(x, t) = \phi_s(x) + \phi_p(x, t) \tag{4.2}$$

where $\phi_p(x, t)$ is a perturbation such that $|\phi_p| \ll 1$. On substituting (4.2) in (2.1) we obtain

$$\phi_{p,xx} - \phi_{p,tt} = \phi_p 2\eta^2 \cosh 4\phi_s - 4\eta \cosh 2\phi_s. \tag{4.3}$$

Now consider solutions of the form

$$\phi_p = f(x)\, e^{\lambda t}. \tag{4.4}$$

This leads to the Schrödinger eigenvalue problem,

$$[-(\partial^2/\partial x^2)+V''(\phi_s)+\lambda^2]f=0 \tag{4.5}$$

for the potential

$$V_0(\phi_s)=2\eta^2\cosh 4\phi_s-4\eta\cosh 2\phi_s-(2\eta^2-4\eta). \tag{4.6}$$

$V_0(\phi_s)$is smooth and bounded and tends to zero as $x\to\pm\infty$. Thus, there exists at most a finite number of bound product solutions for which $|f|\to 0$ as $x\to\pm\infty$. But corresponding to the eigenvalue $\lambda=0$, there exists a non-zero eigenfunction f as

$$f(x,0)=\partial\phi_s/\partial x. \tag{4.7}$$

The nodes of f are infinitely separated; so $\lambda=0$ is the lowest eigenvalue (Morse and Feshbach 1953). This proves the linear stability (Jackiw 1977) of the solution (2.13). The solitary waves obtained by the base equation method as well as by Behera and Khare can be shown by similar arguments to be linearly stable.

5. Asymptotic behaviour of N-solitary wave solution

The N-solitary wave solutions in more than two dimensions are of the form

$$\phi_N=\sinh^{-1}(u_N\{1-\tfrac{1}{2}[(1-2\eta)/(4-2\eta)]u_N^2+(4-2\eta)^{-2}u_N^4\}^{-1/2}) \tag{5.1}$$

where

$$u_N=\sum_{i=1}^{N}a_i\exp(\alpha_i k_i x). \tag{5.2}$$

This can be seen to break up into N simple waves in the asymptotic regions. For as

$$\alpha_i k_i x\to-\infty, \qquad \phi_N\to\sinh^{-1}u_N$$

and as $\alpha_i k_i x\to+\infty$, the dominant term in the braces of (5.1) is $(4-2\eta)^{-2}u_N^4$, consequently

$$\phi_N\approx\sinh^{-1}[(4-2\eta)u_N^{-1}] \qquad \text{as } \alpha_i k_i x\to+\infty.$$

To calculate the phase shift we consider the ith wave in the asymptotic regions:

$$\phi_i=\sinh^{-1}[a_i\exp(\alpha_i k_i x)] \qquad \text{as } x\to-\infty \tag{5.3}$$

$$\phi_i\approx\sinh^{-1}[(4-2\eta)a_i^{-1}\exp(-\alpha_i k_i x)] \quad \text{as } x\to+\infty. \tag{5.4}$$

Defining the corresponding phases (Witham 1974) as

$$\delta_-^i=\log a_i \tag{5.5}$$

$$\delta_+^i=\log[(4-2\eta)a_i^{-1}] \tag{5.6}$$

the phase shift for the ith wave is then given by

$$\Delta_i=\delta_+^i-\delta_-^i\approx\log[(4-2\eta)/a_i^2] \tag{5.7}$$

The N-solitary wave solutions behave as if they were simple waves both at $-\infty$ and $+\infty$, and each component wave nearly undergoes a phase shift given by (5.7). However, there is no loss of stability for the N-solitary wave profile as a whole in the asymptotic regions. Even though the N-solitary wave solutions possess very good stability properties, Derrick's theorem (Derrick 1964) does not permit them to possess finite energy.

6. Topological charges

The conserved topological charge Q associated with a solitary wave in $(1+1)$ dimensions is defined as

$$Q = \int_{-\infty}^{\infty} J^0 \, \mathrm{d}x \tag{6.1}$$

where

$$J^{\mu} = {}^{\mu\upsilon}\partial_{\upsilon}\phi \tag{6.2}$$

and

$$\varepsilon^{01} = 1 \qquad \varepsilon^{\mu\upsilon} = -\varepsilon^{\upsilon\mu}.$$

The Behera–Khare kink can be shown to possess a topological charge

$$Q = 2 \tanh^{-1}[(2-\eta)/(2+\eta)]^{1/2} \qquad |\eta| < 2. \tag{6.3}$$

However, the solitary wave solutions reported herein by us are associated with vanishing topological charge and are, therefore, non-topological configurations (Lee 1976).

7. Conclusion

We have obtained two new classes of solutions for the recently suggested double sinh-Gordon system which is a member of the sine-Gordon family of equations. These solutions possess nice stability and asymptotic properties, but zero topological charge. Whether the asymptotically vanishing non-topological solutions are solitons or not is not clear from the present analytical study. The N-solitary wave solution in more than one space dimension has been shown to break up into N simple waves in the asymptotic regions; similar behaviour has been noted (Witham 1974, Zabusky and Kruskal 1965) for KdV solitons in one space dimension.

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